Convergence of Symmetric Diffusions on Wiener Spaces

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1 Introduction and Framework

Let (H, E, μ) be an abstract Wiener space in the sense of Gross, i.e., E is a real separable Banach space, $H \subset E$ is a continuously and densely embedded real separable Hilbert space, and μ is a centered Gaussian measure on $(E, \mathcal{B}(E))$ with covariance space H. The underlying L^p-space is denoted by $L^p(X, \mu)$. Define $\mathcal{FC}^\infty$ as the set of all functions f of the form

$$f(x) = \varphi(l_1(x)), \ldots, l_n(x)),\ \varphi \in C_b^\infty(R^n),\ l_i \in E^* \tag{1}$$

where $C_b^\infty(R^n)$ is the space of all bounded, infinitely differentiable real-valued functions with bounded partial derivatives, E^* stands for the dual space of E.

For $f \in \mathcal{FC}^\infty$, denote by $\nabla f(x)$ the gradient of f in H, i.e.,

$$< \nabla f(x), h >_H = \left. \frac{df(x + \varepsilon h)}{d\varepsilon} \right|_{\varepsilon=0}, \quad \text{for all} \quad h \in H, \tag{2}$$

where $< \cdot, \cdot >_H$ stands for the inner product in H.

For $p \geq 1$, denote by D_p^1 the completion of $\mathcal{FC}^\infty$ under the norm

$$||f||_{p,1}^p = \int_E |f(x)|^p \mu(dx) + \int_E |\nabla f|_H^p(x) \mu(dx) \tag{3}$$

Take $\phi \in D_{2+\varepsilon}^1$ for some $\varepsilon > 0$ with $\phi > 0$, $a.s.$ and $\int_E \phi^2(x)\mu(dx) = 1$. Consider the symmetric form:

$$\mathcal{E}_\phi^0(f, g) = \int_E < \nabla f(x), \nabla g(x) >_H \ \phi^2(x)\mu(dx), \quad f, g \in \mathcal{FC}^\infty \tag{4}$$

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It is known (see e.g. [RZ1]) that $(\mathcal{E}_\phi^0, D(\mathcal{E}_\phi^0))$ is closable on $L^2(E, \phi^2 d\mu)$, whose closure, denoted by $(\mathcal{E}_\phi, D(\mathcal{E}_\phi))$, is a Dirichlet form. The closability of the form $\mathcal{E}_\phi$ is also equivalent to the closability of the gradient operator ∇ in $L^2(E, \phi^2\mu)$. The action of the closure ∇ on $f \in D(\mathcal{E}_\phi)$ will also be denoted by ∇f.
Let $\Omega = C([0,\infty) \to E)$ be the space of all continuous functions from $[0,\infty)$ into E and let X_t be the coordinate function on Ω such that $X_t(\omega) = \omega(t)$. The diffusion process associated with $(\mathcal{E}_\phi, D(\mathcal{E}_\phi))$ will be denoted by $\{\Omega, X_t, \mathcal{F}_t, P_x, x \in E\}$. The Ornstein -Ulenbeck process associated with the Dirichlet form $\mathcal{E} \equiv \mathcal{E}_1$ will be denoted by $\{\Omega, X_t, \mathcal{F}_t, Q_x, x \in E\}$. Define two probability measures on Ω by

$$P_\phi(\cdot) = \int_E P_x(\cdot)\phi^2\, d\mu, \qquad Q_\mu = \int_E Q_x(\cdot)\, d\mu \tag{5}$$

Let $\{\phi_n, n \geq 1\}$, $\phi_n \in D^1_{2+\varepsilon}$, be a sequence of positive (a.s. with repect to μ) functions such that $\phi_n \to \phi$ in the Sobolev space D^1_2. Denote the diffusion process associated with the Dirichlet form $(\mathcal{E}_{\phi_n}, D(\mathcal{E}_{\phi_n}))$ by $\{\Omega, X_t, \mathcal{F}_t, P^n_x, x \in E\}$. Define

$$P_{\phi_n}(\cdot) = \int_E P_x(\cdot)\phi_n^2\, d\mu$$

In this paper we will show that

$$\sup_{A \in \mathcal{F}_t} |P_{\phi_n}(A) - P_\phi(A)| \to 0\,, \quad t > 0\,,$$

i.e. P_{ϕ_n} converges to P_ϕ in total variation norm on $\mathcal{F}_t$ for any $t > 0$. This convergence is strictly stronger than weak convergence on the full Borel σ-algebra of the path space Ω.
Note that contrarily to previously known results the drifts of the diffusion processes can be very singular, for example where $\phi_n = 0$, $\phi = 0$. The idea of our proof, following the strategy adopted in the finite dimensional situation (see [P]), is to use stopping times arguments to localize the diffusions in some "good" sets where the drifts are sufficiently regular to prove convergence by Girsanov transform, and then to show, by capacity arguments, that the limit diffusion does not hit the "bad" sets. Such a stategy is inspired by Lemma 11.1.1 in [SV]. However there (see [SV], Theorem 11.1.4) the diffusions are then simply localized on increasing bounded balls, whereas we localize on the sets where the ϕ's are uniformly bounded form above and away from zero and where $\phi_n \to \phi$ uniformly along a subsequence.

2 Main Results

Lemma 2.1. There exists a standard Brownian motion $B_t, t \geq 0$ taking values in the Wiener space E such that

$$X_t = X_0 + B_t - \int_0^t X_s ds + 2\int_0^t \frac{\nabla\phi}{\phi}(X_s) ds, \quad P_\phi - a.e. \tag{6}$$

Proof. Define

$$B_t := X_t - X_0 + \int_0^t X_s ds - 2\int_0^t \frac{\nabla\phi}{\phi}(X_s)ds$$

For $l \in E^*$, it follows by integration by parts that

$$\int_E \frac{\partial f}{\partial l}\phi^2 \, d\mu = -\int_E \left(2\frac{\partial\phi}{\partial l}\frac{1}{\phi} - l(x)\right) f\phi^2 \, d\mu$$

Thus, by Fukushima's decomposition (see e.g. [FOT]), $l(B_t)$ is a continuous $\mathcal{F}_t$-martingale with $< l(B), k(B) >_t = < l, k >_H t$ for $l, k \in E^*$, where $<,>$ denotes the sharp bracket of two martingales. Therefore, $B_t, t \geq 0$ is a E-valued Brownain motion following Levy's characterization. The Lemma is proven.

Lemma 2.2. P_ϕ is absolutely continuous with respect to Q_μ on $\mathcal{F}_t$ for $t > 0$.

Proof. Without loss of generality, assume $t = 1$. Suppose first

$$\int_0^1 \left|\frac{\nabla\phi}{\phi}\right|(X_s)ds \leq n, \quad P_\phi - a.e..$$

Define a new probability measure $\bar{Q}_\mu$ on $(\Omega, \mathcal{F}_1)$ by

$$\frac{d\bar{Q}_\mu}{dP_\phi} = \exp\left(-2\int_0^1 \frac{\nabla\phi}{\phi}(X_s)dB_s - 2\int_0^1 \left|\frac{\nabla\phi}{\phi}\right|^2 (X_s)ds\right)$$

By the Girsanov Theorem, we see that

$$X_t = X_0 + \bar{B}_t - \int_0^t X_s ds,$$

where $\bar{B}_t = B_t + 2\int_0^t \frac{\nabla\phi}{\phi}(X_s)ds$ is a Brownian motion under $\bar{Q}_\mu$. It follows from the uniqueness of the Ornstein-Ulenbeck process that $\bar{Q}_\mu = Q_\mu$. So in this case, Q_μ is equivalent to P_ϕ. In the general case, introduce

$$\tau_n = \inf\left\{t \geq 0, \int_0^t \left|\frac{\nabla\phi}{\phi}\right|^2 (X_s)ds > n\right\}$$

Since

$$E_\phi\left[\int_0^t \left|\frac{\nabla\phi}{\phi}\right|^2 (X_s)ds\right] = t\int_E |\nabla\phi|^2(x)d\mu < \infty, \quad t \geq 0\,,$$

it follows that $\tau_n \to \infty$ $P_\phi - a.e.$ as $n \to \infty$. Now, if $Q_\mu(A) = 0$, by the above discussion $P_\phi(A, \tau_n > 1) = 0$. Therefore,

$$P_\phi(A) = P_\phi(A, \tau_n > 1) + P_\phi(A, \tau_n \leq 1) = P_\phi(A, \tau_n \leq 1) \leq P_\phi(\tau_n \leq 1)$$

Letting $n \to \infty$ we get $P_\phi(A) = 0$, which proves the Lemma.

Let $Cap(\cdot)$, $Cap_\phi(\cdot)$ denote the 1-capacities associated with the Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ and $(\mathcal{E}_\phi, D(\mathcal{E}))$ (see [FOT] for details about capacities).

Corollary 2.3. $Cap(F_n) \to 0$ implies $Cap_\phi(F_n) \to 0$.

Proof. This follows from Lemma 2.2 and the probabilistic characterization of capacities in [FOT].

In particular, we have that $Cap_\phi(\phi^2 \geq n) \to 0$ as $n \to \infty$ since the same is true for $Cap(\cdot)$. Moreover, we have

Lemma 2.4.

$$\lim_{n\to\infty} Cap_\phi\left(\phi \leq \frac{1}{n}\right) = 0\,.$$

Proof. It was proven in [RZ1, RZ2] that the Markov uniqueness holds for the Dirichlet form $(\mathcal{E}_\phi, D(\mathcal{E}_\phi))$, which particularly implies that

$$\frac{1}{\phi \vee \frac{1}{n}} \in D(\mathcal{E}_\phi) \qquad \text{for any} \quad n \geq 1$$

By the definition of capacity,

$$\begin{aligned} Cap_\phi\left(\phi \leq \frac{1}{n}\right) &= Cap_\phi\left(\phi \vee \frac{1}{n} \leq \frac{1}{n}\right) \\ &= Cap_\phi\left(\frac{1}{\phi \vee \frac{1}{n}} \geq n\right) \\ &\leq \frac{1}{n^2}\mathcal{E}_{\phi,1}\left(\frac{1}{\phi \vee \frac{1}{n}}, \frac{1}{\phi \vee \frac{1}{n}}\right) \end{aligned}$$

$$= \frac{1}{n^2}\left[\int_E \left(\frac{1}{\phi \vee \frac{1}{n}}\right)^2 (x)\phi^2(x)\mu(dx) + \int_E \left|\nabla \frac{1}{\phi \vee \frac{1}{n}}\right|^2 \phi^2(x)\mu(dx)\right]$$

$$\equiv I^n + II^n.$$

It is clear that

$$\lim_{n\to\infty} I^n = \lim_{n\to\infty} \frac{1}{n^2}\int_E \left(\frac{1}{\phi \vee \frac{1}{n}}\right)^2 (x)\phi^2(x)\mu(dx) = 0 \tag{7}$$

Now,

$$\begin{aligned} \nabla \frac{1}{\phi \vee \frac{1}{n}} &= \left(\frac{1}{\phi \vee \frac{1}{n}}\right)^2 \nabla\left(\phi \vee \frac{1}{n}\right) \\ &= \left(\frac{1}{\phi \vee \frac{1}{n}}\right)^2 \left[\nabla\phi|_{\{\phi > \frac{1}{n}\}} + \frac{1}{2}\nabla\phi|_{\{\phi = \frac{1}{n}\}}\right] \end{aligned}$$

It follows that

$$
\begin{aligned}
II^n &\le \frac{1}{n^2}\int_{\{\phi\ge\frac{1}{n}\}}\left(\frac{1}{\phi\vee\frac{1}{n}}\right)^4(x)|\nabla\phi|^2(x)\phi^2(x)\mu(dx)\\
&\le \frac{1}{n^2}\int_{\{\phi\ge\frac{1}{n}\}}\left(\frac{1}{\phi\vee\frac{1}{n}}\right)^2(x)|\nabla\phi|^2(x)\mu(dx)
\end{aligned}
\tag{8}
$$

Observe that

$$\frac{1}{n^2}|_{\{\phi\ge\frac{1}{n}\}}\left(\frac{1}{\phi\vee\frac{1}{n}}\right)^2|\nabla\phi|^2\to 0\quad a.s.$$

as $n\to\infty$ and

$$\frac{1}{n^2}|_{\{\phi\ge\frac{1}{n}\}}\left(\frac{1}{\phi\vee\frac{1}{n}}\right)^2|\nabla\phi|^2\le|\nabla\phi|^2\in L^1(E,\mu)$$

By dominated convergence theorem, we have

$$\lim_{n\to\infty}II^n=0$$

Hence,

$$\lim_{n\to\infty}Cap_\phi\left(\phi\le\frac{1}{n}\right)=0\,.$$

Let $\phi_n, n\ge 1$ be a sequence functions in D_2^1 with $\phi_n>0$ $a.s.$.

Lemma 2.5. Assume $\phi_n\to\phi$ in the Sobolev space D_2^1 as $n\to\infty$. Then there exists a decreasing sequence $\{G_m, m\ge 1\}$ of open subsets of E satisfying
(i)

$$\lim_{m\to\infty}Cap_\phi(G_m)=0\,;$$

(ii)

$$\frac{1}{m}\le\phi\le m\quad\text{on } G_m^c\,;$$

(iii) There exists a subsequence $\{\phi_{n_k}, k\ge 1\}$ such that

$$\lim_{k\to\infty}\phi_{n_k}=\phi$$

uniformly on G_m^c for each m.

Proof. By Lemma 2.4, there exists a decreasing sequence $\{G'_m, m\ge 1\}$ of open subsets such that $Cap_\phi(G'_m)\to 0$ and $\frac{1}{m}\le\phi\le m$ on $(G'_m)^c$. Since $\phi_n\to\phi$ in D_2^1, by Theorem 2.1.4 in [FOT] one can find a decreasing sequence $\{G''_m, m\ge 1\}$ of open subsets such that $Cap(G''_m)\to 0$ as $m\to\infty$ and $\phi_n\to\phi$ along a subsequence, uniformly on $(G''_m)^c$ for each m. By Corollary 2.3, $Cap_\phi(G''_m)\to 0$ as $m\to\infty$. Now, set $G_m=G'_m\cup G''_m$. Then $\{G_m, m\ge 1\}$ are the open subsets desired.

Define

$$\tau_m = \inf\{t > 0; X_t \in G_m\} \tag{9}$$

It follows from (i) in Lemma 2.5 and the probabilistic characterization of the capacity in [FOT] that

$$\lim_{m\to\infty} P_\phi(\tau_m < t) = 0. \tag{10}$$

Theorem 2.6. Assume that $\phi_n, \phi \in D^1_{2+\varepsilon}$ for some positive constant ε and that $\phi_n \to \phi$ in the Sobolev space D^1_2 as $n \to \infty$. Then

$$\sup_{A\in\mathcal{F}_t} |P_{\phi_n}(A) - P_\phi(A)| \to 0$$

for any $t > 0$.

Proof. Define for $m \geq 1$,

$$\psi^{n,m} = \frac{1}{m} \vee \phi_n \wedge m \qquad \psi^m = \frac{1}{m} \vee \phi \wedge m$$

Denote respectively by $\{\Omega, X_t, \mathcal{F}_t, P^{(n,m)}_x, x \in E\}$ and $\{\Omega, X_t, \mathcal{F}_t, P^{(m)}_x, x \in E\}$ the diffusion processes associated with the Dirichlet forms $(\mathcal{E}_{\psi^{n,m}}, D(\mathcal{E}_{\psi^{n,m}}))$ and $(\mathcal{E}_{\psi^m}, D(\mathcal{E}_{\psi^m}))$ defined as in (4) with ϕ replaced by $\psi^{n,m}$ and ψ^m. Set,

$$P^{(n,m)} = \int_E P^{(n,m)}_x(\cdot)(\psi^{n,m})^2 d\mu, \quad P^{(m)} = \int_E P^{(m)}_x(\cdot)(\psi^m)^2 d\mu\,. \tag{11}$$

For brevity let us denote by $\{\phi_k, k \geq 1\}$ the subsequence given in Lemma 2.5(iii) and let $P^{(k,m)}$ the corresponding measures. Note that, for k suffciently large,

$$P^{(k,m+2)}|_{\mathcal{F}_{\tau_m}} = P_{\phi_k}|_{\mathcal{F}_{\tau_m}}, \qquad P^{(m+2)}|_{\mathcal{F}_{\tau_m}} = P_\phi|_{\mathcal{F}_{\tau_m}}\,, \tag{12}$$

where "$|_{\mathcal{F}_{\tau_m}}$" stands for the restriction of the corresponding measure on the σ-field $\mathcal{F}_{\tau_m}$, By [ARZ], Theorem 1.3 and Remark 3.4, $P^{(n,m+2)} \sim Q_\mu$ and $P^{(m+2)} \sim Q_\mu$ on $\mathcal{F}_t$ for any $t > 0$, with

$$\left.\frac{dP^{(n,m+2)}}{dQ_\mu}\right|_{\mathcal{F}_t} = L^{\psi^{n,m+2}}_t, \quad \left.\frac{dQ_\mu}{dP^{(n,m+2)}}\right|_{\mathcal{F}_t} = L^{1/\psi^{n,m+2}}_t, \tag{13}$$

and

$$\left.\frac{dP^{(m+2)}}{dQ_\mu}\right|_{\mathcal{F}_t} = L^{\psi^{m+2}}_t, \quad \left.\frac{dQ_\mu}{dP^{(m+2)}}\right|_{\mathcal{F}_t} = L^{1/\psi^{m+2}}_t, \tag{14}$$

where

$$L^\psi_t := \exp\left(M^{\ln\psi}_t - \frac{1}{2}\int_0^t \left|\frac{\nabla\psi}{\psi}\right|^2 (X_s) ds\right),$$

and $M^{\ln\psi}_t$ denotes the martingale additive functional parts in the Fukushima's decomposition of the Dirichlet processes $\ln\psi(X_t) - \ln\psi(X_0)$, see [FOT].

Let us denote by $\widetilde{P}^{(n,m+2)}$ and $\widetilde{P}^{(m+2)}$ the probability measures defined by

$$\left.\frac{d\widetilde{P}^{(n,m+2)}}{dQ_\mu}\right|_{\mathcal{F}_t} := L^{\psi^{n,m+2}}_{t\wedge\tau_m}, \quad \left.\frac{d\widetilde{P}^{(m+2)}}{dQ_\mu}\right|_{\mathcal{F}_t} := L^{1/\psi^{m+2}}_{t\wedge\tau_m}. \tag{15}$$

Hence

$$\widetilde{P}^{(k,m+2)}|_{\mathcal{F}_{\tau_m}} = P^{(k,m+2)}|_{\mathcal{F}_{\tau_m}} = P_{\phi_k}|_{\mathcal{F}_{\tau_m}}, \tag{16}$$

$$\widetilde{P}^{(m+2)}|_{\mathcal{F}_{\tau_m}} = P^{(m+2)}|_{\mathcal{F}_{\tau_m}} = P_{\phi}|_{\mathcal{F}_{\tau_m}}, \tag{17}$$

$$\widetilde{P}^{(n,m+2)} \sim \widetilde{P}^{(m+2)} \quad \text{on } \mathcal{F}_t \text{ for any } t>0, \tag{18}$$

and

$$\begin{aligned}\left.\frac{dP^{(n,m+2)}}{dP^{(m+2)}}\right|_{\mathcal{F}_t} &= \exp\Big(M^{\ln\psi,m+2}_{t\wedge\tau_m} - M^{-\ln\psi^{m+2}}_{t\wedge\tau_m}\\ &-\frac{1}{2}\int_0^{t\wedge\tau_m}\left(\left|\frac{\nabla\psi^{n,m+2}}{\psi^{n,m+2}}\right|^2 - \left|\frac{\nabla\psi^{m+2}}{\psi^{m+2}}\right|^2\right)(X_s)ds\Big).\end{aligned} \tag{19}$$

Proceeding as in the proof of Lemma 3.1 in [DP] (a sort of versions in total variation norm of Lemma 11.1.1 in [SV]) one easily gets

$$\sup_{A\in\mathcal{F}_t}|P_{\phi_k}(A)-P_\phi(A)| \le 3\sup_{A\in\mathcal{F}_t}|\widetilde{P}^{(k,m+2)}(A)-\widetilde{P}^{(m+2)}(A)|+4P_\phi(\tau_m<t) \tag{20}$$

and, by the Csizlár-Kullback inequality and (19), one obtains

$$\begin{aligned}&\left(\sup_{A\in\mathcal{F}_t}|\widetilde{P}^{(k,m+2)}(A)-\widetilde{P}^{(m+2)}(A)|\right)^2 \le 2E_{\widetilde{P}^{(k,m+2)}}\left[\ln\left.\frac{d\widetilde{P}^{(k,m+2)}}{d\widetilde{P}^{(m+2)}}\right|_{\mathcal{F}_t}\right]\\ =\;& E_{\widetilde{P}^{(k,m+2)}}\left[\int_0^{t\wedge\tau_m}\left(\left|\frac{\nabla\psi^{k,m+2}}{\psi^{k,m+2}}\right|^2-\left|\frac{\nabla\psi^{m+2}}{\psi^{m+2}}\right|^2\right)(X_s)ds\right]\\ =\;& \int_0^t E_{P_{\phi_k}}\left[1_{\{\tau_m>s\}}\left(\left|\frac{\nabla\psi^{k,m+2}}{\psi^{k,m+2}}\right|^2-\left|\frac{\nabla\psi^{m+2}}{\psi^{m+2}}\right|^2\right)(X_s)\right]ds\\ \le\;& t\|\nabla\phi_k-\nabla\phi\|^2_{L^2(E,\mu)}+tm^2\|\nabla\phi\|^2_{L^2(E,\mu)}\|\phi_k-\phi\|^2_{L^\infty(G^c_m,\mu)}.\end{aligned}$$

Thus by Lemma 2.5(iii), (10) and (20) $\sup_{A\in\mathcal{F}_t}|P_{\phi_n}(A)-P_\phi(A)|$ converges along a subsequence. Suppose now that the whole sequence does not converge. Then there exists a subsequence $\{P_{\phi_{n_j}}, j\ge 1\}$ such that

$$\sup_{A\in\mathcal{F}_t}|P_{\phi_{n_j}}(A)-P_\phi(A)|>\varepsilon, \qquad \text{for any } j.$$

Since $\phi_{n_j}\to\phi$ in D^1_2, we get a contradiction. This completes the proof.

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